This note summarize some useful facts for Math 215 final exam.

Fact 1. The area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$ is $ab\pi$

* This fact can be used on the exam without proof.

- Explanation :

Let D be the domain given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

Set $u=\frac{x}{a}$, $v=\frac{y}{b}$ $\Rightarrow$ D is given by $u^2+v^2\le 1$.

$$x=ax,\ y=bv \Rightarrow \underbrace{J(u,v)}_{\text{Jacobian}} = \left|\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\right| = \left|\det\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\right| = ab$$

$$\text{Area}(D)=\iint_D 1\,dxdy=\iint_{u^2+v^2\le 1} 1\cdot\underbrace{ab}_{\text{Jacobian}}\,dudv = ab\underbrace{\iint_{u^2+v^2\le 1} 1\,dudv}_{\text{Area of unit disk}} = ab\pi$$

Fact 2. The outward unit normal vector of the sphere $x^2+y^2+z^2=R^2$ is given by $\vec{n}=\left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right)$

* This fact can be used on the exam without proof.

- Explanation :

The sphere $x^2+y^2+z^2$ is a level surface of $f(x,y,z)=x^2+y^2+z^2$.

A normal vector is $\nabla f=(2x,2y,2z)$

The unit normal vector is

$$\vec{n}=\frac{\nabla f}{|\nabla f|}=\frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}}=\frac{(2x,2y,2z)}{\sqrt{4R^2}}=\frac{(2x,2y,2z)}{2R}=\left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right)$$

which is oriented outward (pointing away from the origin)

Fact 3. The vortex field $\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ satiesfies the following properties:

(1) $P = -\frac{y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(2) If C is a circle centered at $(0,0)$ with counterclockwise orientation, then $\int_C \vec{V} \cdot d\vec{r} = 2\pi$.

(3) If C is a simple loop which does not enclose $(0,0)$, then $\int_C \vec{V} \cdot d\vec{r} = 0$.

(4) If C is a simple loop which encloses $(0,0)$ with counterclockwise orientation, then $\int_C \vec{V} \cdot d\vec{r} = 2\pi$.

* This fact cannot be used without justification.

- Explanation of (1):

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = -\frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- Explanation of (2):

Let R be the radius of C

$\Rightarrow C$ is parametrized by $\vec{r}(t) = (R\cos t, R\sin t)$ with $0 \le t \le 2\pi$.

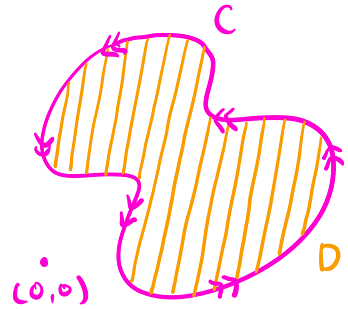
$$\int_C \vec{V} \cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$\vec{V}(\vec{r}(t)) = \left(-\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2}\right), \quad \vec{r}'(t) = (-R\sin t, R\cos t)$$

$$\Rightarrow \vec{V}(\vec{r}(t)) \cdot \vec{r}'(t) = -\frac{R\sin t}{R^2}\cdot(-R\sin t) + \frac{R\cos t}{R^2}\cdot R\cos t = \sin^2 t + \cos^2 t = 1.$$

$$\Rightarrow \int_C \vec{V} \cdot d\vec{r} = \int_0^{2\pi} 1\,dt = 2\pi.$$

- Explanation of (3):



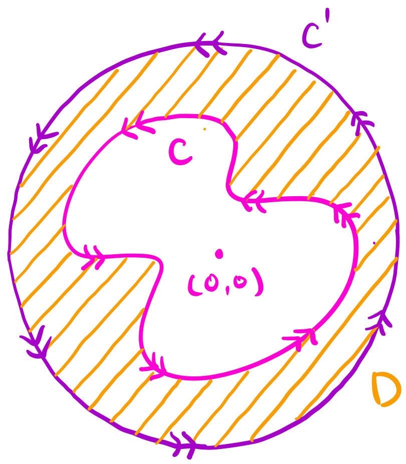
D : the region enclosed by C

$\Rightarrow \partial D = C$ is positively oriented.

$\vec{V}$ is defined on D (D does not contain $(0,0)$).

$$\int_C \vec{V}\cdot d\vec{r} = \int_{\partial D} \vec{V}\cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \underset{(1)}{=} 0.$$

- Explanation of (4):



C': a circle centered at $(0,0)$ which encloses C with counterclockwise orientation

D : the region bounded by C and C'.

$\Rightarrow \partial D = -C + C'$ is positively oriented.

(C is negatively oriented)

$\vec{V}$ is defined on D (D does not contain $(0,0)$)

$$\int_{\partial D} \vec{V}\cdot d\vec{r} = -\int_C \vec{V}\cdot d\vec{r} + \int_{C'} \vec{V}\cdot d\vec{r}$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = -\int_{\partial D} \vec{V}\cdot d\vec{r} + \int_{C'} \vec{V}\cdot d\vec{r} \quad (*)$$

$$\int_{\partial D} \vec{V}\cdot d\vec{r} \underset{\text{Green's thm}}{=} \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \underset{(1)}{=} 0.$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} \underset{(*)}{=} \int_{C'} \vec{V}\cdot d\vec{r} \underset{(2)}{=} 2\pi$$

Note You cannot consider the region bounded by C because $\vec{V}$ is not defined at $(0,0)$.

Fact 4 The inverse square field

$$\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

satisfies the following properties:

(1) $\mathrm{div}(\vec{F}) = 0$

(2) If S is a sphere centered at $(0,0,0)$ with outward orientation, then $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$.

(3) If S is a boundary surface which does not enclose $(0,0,0)$, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

(4) If S is a boundary surface which encloses $(0,0,0)$ with outward orientation, then $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$.

* This fact cannot be used without justification.

- Explanation of (1):

$$P = \frac{x}{(x^2+y^2+z^2)^{3/2}}, \quad Q = \frac{y}{(x^2+y^2+z^2)^{3/2}}, \quad R = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$P_x = \frac{1 \cdot (x^2+y^2+z^2)^{3/2} - x \cdot \frac{3}{2}(x^2+y^2+z^2)^{1/2} \cdot 2x}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2-3x^2)}{(x^2+y^2+z^2)^3} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, we get $Q_y = \dfrac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$ and $R_z = \dfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

(We get Q_y from P_x by swapping x and y
We get R_z from P_x by swapping x and z)

$$\Rightarrow \mathrm{div}(\vec{F}) = P_x + Q_y + R_z = \frac{(y^2+z^2-2x^2)+(x^2+z^2-2y^2)+(x^2+y^2-2z^2)}{(x^2+y^2+z^2)^{5/2}} = 0.$$

- Explanation of (2):

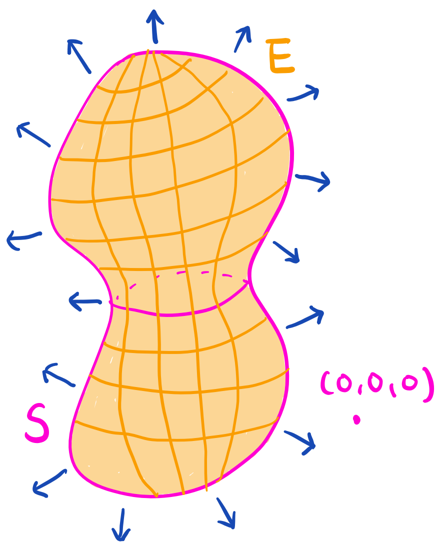
  Let R be the radius of S.

  The unit normal vector of S is $\vec{n} = \left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right)$ (Fact 2)

$$\vec{F}\cdot\vec{n} = \frac{x^2}{R(x^2+y^2+z^2)^{3/2}} + \frac{y^2}{R(x^2+y^2+z^2)^{3/2}} + \frac{z^2}{R(x^2+y^2+z^2)^{3/2}} = \frac{x^2+y^2+z^2}{R(x^2+y^2+z^2)^{3/2}} \underset{\substack{\uparrow \\ x^2+y^2+z^2=R^2 \text{ on } S}}{=} \frac{R^2}{R\cdot R^3} = \frac{1}{R^2}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S \frac{1}{R^2}\, dS = \frac{1}{R^2}\text{Area}(S) = \frac{1}{R^2}\cdot \underbrace{4\pi R^2}_{\text{Area of sphere}} = 4\pi$$

- Explanation of (3):



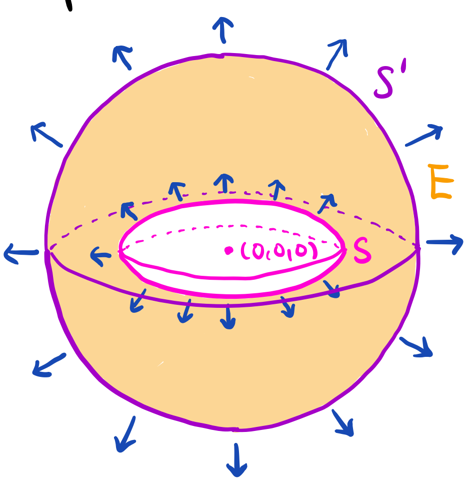
E: the solid bounded by S.

$\Rightarrow \partial E = S$ is oriented outward.

$\vec{F}$ is defined on E (E does not contain $(0,0,0)$)

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \text{div}(\vec{F})\, dV \underset{\substack{\uparrow \\ (1)}}{=} 0$$

- Explanation of (4):



S': a sphere centered at $(0,0,0)$ which encloses S with outward orientation.

E: the solid bounded by S and S'.

$\Rightarrow \partial E = -S + S'$ is oriented outward.

(S is oriented inward with respect to E)

$\vec{F}$ is defined on E (E does not contain $(0,0,0)$)

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S} \quad (*)$$

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \text{div}(\vec{F})\, dV \underset{\substack{\uparrow \\ (1)}}{=} 0 \Rightarrow \iint_S \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ (*)}}{=} \iint_{S'} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ (2)}}{=} 4\pi$$